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Intrinsically localized chaos in discrete nonlinear extended systems

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Abstract. – The phenomenon of intrinsic localization in discrete nonlinear extended systems, *i.e.* the (generic) existence of discrete breathers, is shown to be not restricted to periodic solutions but it also extends to more complex (chaotic) dynamical behaviour. We illustrate this with two different forced and damped systems exhibiting this type of solutions: In an anisotropic Josephson junction ladder, we obtain intrinsically localized chaotic solutions by following periodic rotobreather solutions through a cascade of period-doubling bifurcations. In an array of forced and damped van der Pol oscillators, they are obtained by numerical continuation (path-following) methods from the uncoupled limit, where its existence is trivially ascertained, following the ideas of the anticontinuum limit.

Discrete homogeneous arrays of (hamiltonian and non-hamiltonian) nonlinear oscillators (or rotors) exhibit generic solutions which are time-periodic and (typically exponentially) localized in space. These solutions are called *discrete breathers* by analogy with non-topological localized solutions of certain PDE's. In contrast with continuous "bona fide" breathers, discrete breathers posses a remarkable structural stability, and thus genericity. This localization is often referred to as *intrinsic* to stress the fact that the system is homogeneous (no impurities or disorder are present). For an updated and comprehensive review on discrete breathers, see[1].

A general schematic way to describe a discrete breather in a one-dimensional lattice is the following: Let us consider the phase space Γ_s of a single oscillator (or rotor), so that the phase space Γ of the network is the cartesian product of the single site phase spaces. Let denote by A, B periodic orbits in Γ_s, eventually projections of trajectories of Γ onto Γ_s. A discrete

breather is a solution

$$\{\phi_i(t)\} \equiv \{\ldots, B_{-2}\ , B_{-1}\ , A\ , B_1\ , B_2\ , \ldots\} \tag{1}$$

with $\lim_{|i|\to\infty} B_i = B_\infty$, and $B_\infty \neq A$. Archetypical examples are Klein-Gordon hamiltonian breathers, where A is a periodic cycle of frequency ω_b in the $(\phi, \dot{\phi})$ phase space, B_∞ is the rest solution $(0,0)$, and B_i are ω_b-cycles with exponentially decreasing amplitude. In the case of forced and damped arrays, A and B_∞ are usually ω_b-cycles of different amplitude. If A is an ω_b-cycle non-homotopic to zero (*i. e.* the central oscillator rotates), the term *rotobreather* is used[2, 3].

In this letter we present numerical evidences as well as plausibility arguments strongly supporting the conclusion that the phenomenon of intrinsic localization in discrete nonlinear extended systems is not restricted to time-periodic solutions, but it extends to more complex (chaotic) behaviour in a generic way for damped and forced systems. More specifically, we show below examples of solutions of the type schematized in (1), where A is a *chaotic trajectory*, B_∞ is a "regular" ω_b-cycle, and B_i are "noisy" cycles, with "noise intensity" exponentially decreasing to zero as $|i|$ grows. The first example concerns the operation of a Josephson junction device, the *Josephson junction ladder*, which has recently received some attention from both theoretical[4, 5, 6, 7] and experimental[8] sides, in connection to the relevance of nonlinear dynamics of discrete systems in Condensed Matter Physics. The second example, though also experimentally realizable, serves us to illuminate possible pathways towards a rigorous characterization of the genericity of intrinsically localized chaos in discrete nonlinear extended systems, in the spirit of the ideas of the, so called, *anticontinuum limit*[9, 10, 11] approach to intrinsic localization. We end with a short discussion on the implausibility of existence of this type of solutions as exact ones in hamiltonian arrays. Earlier numerical observations of localized chaotic solutions seems to have been reported in[12](coupled map lattices) and[13](domain walls in a parametrically excited lattice of oscillators). Our results establish a precise (and very general) link between situations of spatio-temporal complex behaviour in spatially extended discrete systems and the emergent new results and powerful methods of intrinsic localization.

Recent theoretical analyses of the dynamics of an anisotropic Josephson junction ladder (see figure 1) with injected ac currents [5] have shown the existence of discrete breathers as attracting solutions of the equations of motion describing the dynamics of the system in the framework of the resistively and capacitively shunted junction (RCSJ) approach [14]. The existence of discrete breathers in Josephson junction arrays should indeed be regarded as generic, given the connection between the general description of these systems in terms of the superconducting Ginzburg-Landau order parameter $\Psi(\vec{x}) = |\Psi(\vec{x})|\exp(i\theta(\vec{x}))$, where $\vec{x}$ denotes the superconducting island position, and the *discrete nonlinear Schrödinger equation*, for the case of ideal (perfect insulating) junctions [5]. In fact, the quantum Hamiltonian of a single ideal Josephson junction corresponds to the problem of two coupled anharmonic quantum oscillators, for which the asymmetric classical breather solutions have been shown to persist in the quantum regime as very long lifetime states [15] (see also [16]). When the energy cost to add an extra Cooper pair on a neutral superconducting island (*charging energy* E_c) is much lower than the tunneling energy (*Josephson energy* E_J) the superconducting phase $\theta(\vec{x})$ becomes a good (very weakly fluctuating) variable to describing the island state, thus validating the RCSJ approach [14]. This is the situation when the superconducting islands are of macroscopic size. The validity of the RCSJ approach in the regime $E_c/E_J \ll 1$ is a well established issue and its predictions fit very well with experiments[18].

θ_i and θ'_i will denote, respectively, the phases of upper and lower islands at site i in the ladder; the currents $I(t) = I_{ac}\cos(\omega t)$ are injected into the islands in the upper row and

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Fig. 1. – Schematic picture of the JJ ladder showing the injection of the currents in the array.

extracted from those in the lower row; $(J_x,\ \epsilon_x)$ are the junction characteristics for junctions in horizontal links and $(J_y,\ \epsilon_y)$ for junctions in vertical links. With the change of variables $\chi_i = \frac{1}{2}(\theta_i + \theta_i')$, $\phi_i = \frac{1}{2}(\theta_i - \theta_i')$, the RCSJ equations [5] are

$$\begin{aligned}\ddot{\chi}_i = \quad & J_x \left[\sin(\chi_{i+1} - \chi_i)\cos(\phi_{i+1} - \phi_i) + \sin(\chi_{i-1} - \chi_i)\cos(\phi_{i-1} - \phi_i)\right] \\ & \qquad + \epsilon_x \left(\dot{\chi}_{i+1} + \dot{\chi}_{i-1} - 2\dot{\chi}_i\right)\end{aligned} \tag{2}$$

$$\begin{aligned}\ddot{\phi}_i = \quad & J_x \left[\cos(\chi_{i+1} - \chi_i)\sin(\phi_{i+1} - \phi_i) + \cos(\chi_{i-1} - \chi_i)\sin(\phi_{i-1} - \phi_i)\right] \\ & \qquad + \epsilon_x \left(\dot{\phi}_{i+1} + \dot{\phi}_{i-1} - 2\dot{\phi}_i\right) - J_y \sin(2\phi_i) - 2\epsilon_y \dot{\phi}_i - I(t)\end{aligned} \tag{3}$$

With uniform initial conditions in the "center of mass" coordinates and momenta: χ_i and $\dot{\chi}_i$ independent of i, equations (2) have the solution $\chi_i(t) = \Omega t + \alpha$ for all i; this effectively decouples equations (3) for the ϕ_i variables from equations (2) for the χ_i variables. Then, using efficient *continuation methods* [10, 11] from the uncoupled (anticontinuum) limit ($J_x = \epsilon_x = 0$), one easily computes discrete breather solutions; these turn out to be attractors of the dynamics of the ladder in a wide range of parameter values.

We will concentrate on the *rotobreather* type of solutions, in which the phase half-difference ϕ_{j^*} through a vertical junction rotates, while the rest ϕ_i ($i \neq j^*$) oscillate, and the "center of mass" variables χ_i remain uniformly at rest ($\Omega = \alpha = 0$; note that any other values for these parameters, fixed by the uniform initial conditions, would show the same behavior). The period of the rotobreather solution is $T_b = 2\pi/\omega_b = 4\pi/\omega$, where ω is the frequency of the external currents.

By performing the Floquet analysis of rotobreather solutions, one can determine the regions of linear stability in parameter space, whose borders correspond to different types of bifurcations[7]. One of them (which occurs typically when varying the external frequency ω) is a period-doubling bifurcation: The (destabilizing) eigenvector of the Floquet matrix, which is associated to the eigenvalue exiting the unit circle (in complex plane) at -1, is (exponentially) *localized* at the center of the rotobreather and then, a new (linearly stable) rotobreather with frequency $\omega_b/2$ exists past the bifurcation. This new rotobreather can be easy and *safely* obtained by slightly perturbing the unstable rotobreather along the direction of the destabilizing eigenvector. In other words, although one cannot continue the localized solution in a bifurcation, local bifurcation analysis helps to throw a bridge over the bifurcation, so arriving safely to the new localized solution at the other side.

Continuously varying the external frequency ω, further period doubling bifurcations are

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Fig. 2. – Lyapunov spectrum of a chaotic rotobreather in the JJ ladder.

often found leading to a chaotic solution. In order to characterize unambiguosly this solution as chaotic, we have computed its *Lyapunov spectrum* $\{\lambda_i\}$, which is shown in figure 2. There is only one positive Lyapunov exponent, $\lambda_1 = 0.049$bits/s. As we are dealing here with a continuous time dynamical system, a null exponent is also present. The rest of the spectrum is negative. Thus, there is only one expanding direction (degree of freedom) in phase space. The estimated Lyapunov dimension, D_L, defined[17] as

$$D_L = j + \frac{1}{|\lambda_{j+1}|} \sum_{i=1}^{j} \lambda_i \qquad (4)$$

with j such that $\sum_{i=1}^{j} \lambda_i > 0$ and $\sum_{i=1}^{j+1} \lambda_i < 0$ (exponents are ordered in decreasing order), is $D_L = 4.7$.

A look at the profile (at different times) of the Lyapunov vector associated with the positive Lyapunov exponent reveals that it is strongly localized in space. As the period doubling bifurcations leading to the chaotic solution are driven by exponentially localized eigenvectors of the Floquet matrix, it is not surprising that this chaotic solution is exponentially localized. In figure 3 we show the Poincaré (stroboscopic, with period $2T_b$) section of the central rotor trajectory $\phi_0(t) \pmod{2\pi}$ of the intrinsically localized chaotic solution for parameter values $J_x = 0.05$, $J_y = 0.5$, $\epsilon_x = 0.03$, $\epsilon_y = 0.01$, $\omega = 1.623$ and $I_{ac} = 0.72$. As shown also in figure 3, the trajectories $\phi_i(t)$ for $|i| > 0$ are noisy (or chaotically perturbed) oscillations. As a rough measure of "noise intensity", we adopt the radius r_i of the smallest circle containing the Poincaré section of the ith oscillator. This quantity decreases exponentially $r_i \simeq C\exp(-|i|/\xi)$ ($\xi \sim 1.13$), as evidenced in figure 3.

Vaguely speaking, one could say that the uniformly oscillating solution is robust enough to exponentially damp out the penetration of the chaotic perturbation produced by the central rotor; equivalently, one could say that the uniformly oscillating solution posses *finite coherence length* ξ, so that an oscillator does not feel the effect of any sustained local perturbation located at distances much greater than ξ (lattice units) from it. On intuitive basis, it is clear that finite coherence length is required for intrinsic localization to occur[19].

Now we turn to the question on genericity, *i. e.* should one expect that these intrinsically localized chaotic solutions exist generically in discrete arrays of coupled nonlinear oscillators? Though arguably there is little doubt that finite coherence length is ubiquitous in discrete nonlinear extended systems, at least some degree of robustness of the chaotic trajectory in the central oscillator (not to speak of the mere possibility of a chaotic behaviour) is also needed. In an attempt to pave the way towards rigorous answers to the question, we have considered the perspective on intrinsic localization opened by the "anticontinuum limit" [9, 10, 11] approach,

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Fig. 3. – Poincaré (stroboscopic) sections of the trajectories $\phi_i(t)$ at times $t+n2T_b$. The pictures show in quite different scales the planes $\dot{\phi}$ (in radians per unit time) vs. ϕ (in radians) for the rotor (a), its fifth neighbour (b) and the ninth one (c). In (d), the logarithm of the "noise amplitude" plotted vs. the neighboring index shows the exponentially localized character of chaos.

as explained below.

Let us consider a chain of forced and damped identical *uncoupled* oscillators, and assume that there is coexistence of a chaotic attractor and an attracting cycle in the single oscillator phase space. Now, consider the cartesian product of a (central site) chaotic attractor and attracting regular cycles in the rest of lattices sites. This set is an attractor in the phase space of the uncoupled chain, which could plausibly be continued when coupling is turned on.

In order to check this idea, we have chosen a chain of harmonically coupled, forced van der Pol oscillators:

$$\ddot{\phi}_i = -\mu(\phi_i^2 - 1)\dot{\phi}_i - \phi_i + b\cos(\omega t) + C(\phi_{i+1} - 2\phi_i + \phi_{i-1}) \tag{5}$$

For $\mu = 4.033$, $b = 9.0$ and $\omega = \pi$, the single forced van der Pol oscillator phase space shows coexistence of two strange attractors and a periodic cycle of frequency $\omega/3$ (see [20]). We have numerically continued the solution of the uncoupled chain in which the central oscillator follows a chaotic trajectory in one of the strange attractors, while the rest of the oscillators follow the periodic cycle, for non-zero values of the coupling constant C, up to values of the order of 0.5×10^{-3}, which are small but significantly different from zero.

The continuation from the uncoupled limit provides a systematic way of obtaining intrinsically localized chaotic solutions, provided the coexistence of strange and periodic attractors for a single oscillator. It may also serve, like in the simpler case of periodic discrete breathers, as a basis for the construction of a proof of existence which we see as a difficult problem. Indeed, Mackay [21] already mentioned this approach for the case of the Plykin attractor, where continuation is ensured due to uniform hyperbolicity; unfortunately, as usual in chaos theory, strong conditions which simplify mathematical proofs do not seem to fit easily into realistic physical models.

The examples we have shown here concern systems of forced and damped oscillators, and one may wonder about hamiltonian arrays of oscillators. Though we do not have a definite

answer on the existence of intrinsic localized chaos in discrete nonlinear Hamiltonian extended systems, it seems plausible that the typical "broad band" structure of the power spectrum of chaotic trajectories would imply a violation of the condition of non-resonance with the phonons [10]. In the extent that this condition plays an essential role in the proof of existence of hamiltonian discrete breathers, we think that the answer is negative. However, chaotic breathers in discrete Hamiltonian arrays easily appear as long-lived transient solutions. An observation of erratically moving transient chaotic breathers in hamiltonian Fermi-Pasta-Ulam chains has been recently reported[22]. After completion of this work, we became aware of the numerical observation of chaotic rotobreathers by Bonart and Page[23] in a 1d driven damped lattice of dipoles.

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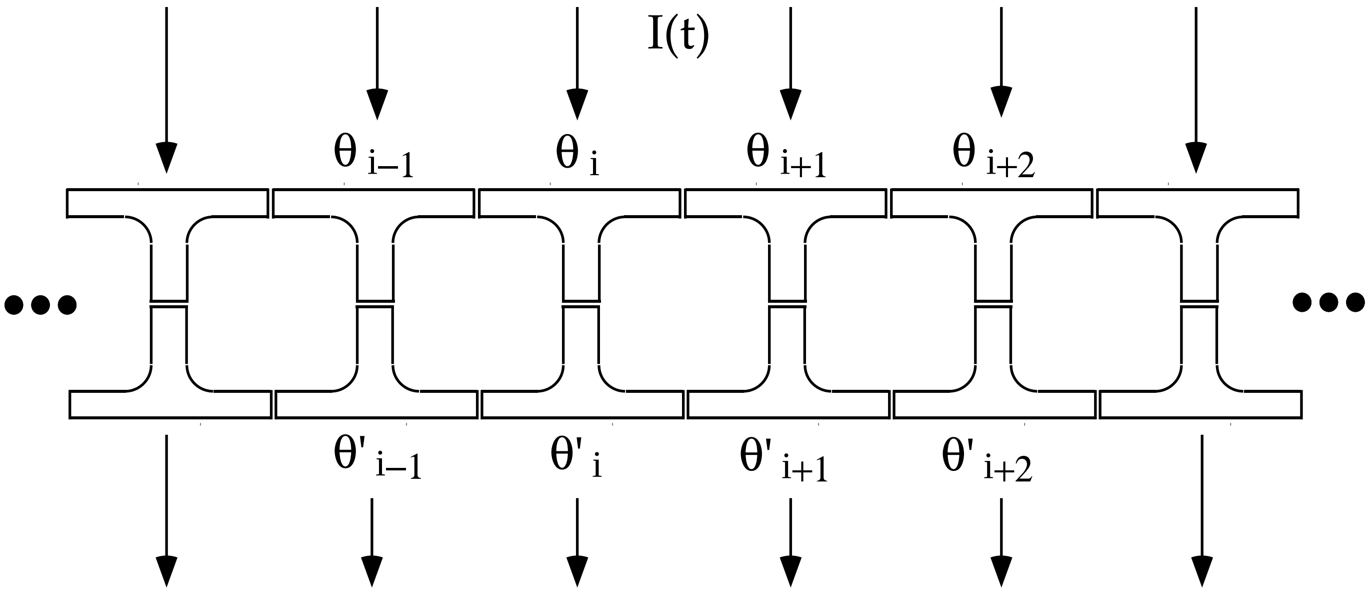
REFERENCES

[1] Flach S. and Willis C. R.,*Phys. Rep.*, **295**(1998) 181 .
[2] Takeno S. and Peyrard M., *Physica D*, **92** (1996)140; *Phys. Rev. E*, **55**(1997) 1922.
[3] Aubry S., in *Workshop on* Lattice Dynamics ,, Paris, June 1995, to appear in *Physica D*, (1997).
[4] Mazo J. J., Falo F. and Floría L. M., *Phys. Rev. B*, **52** (1995)10433.
[5] Floría L. M., Marín J. L., Martínez P. J., Falo F. and Aubry S.,*Europhys. Lett.*, **36**(1996) 539 ; Floría L.M. *et al.*,*Physica D*,**113** (1997)387.
[6] Barahona M., Strogatz S. H. and Orlando T. P.,*Phys. Rev. B*, **57**(1998)1181.
[7] Martínez P. J. *et al.*,*Physica D*, **119** (1998)175.
[8] Barahona M., Trías E., Orlando T. P., Duwell A. E., Van der Zant H. S. J., Watanabe S. and Strogatz S. H., *Phys. Rev. B*, **55**(1997)11989; Van der Zant H. S. J. *et al.*,*Physica D*,**119**(1998) 219.
[9] Aubry S.,*Physica D*, **71**(1994) 196; **86**(1995)284 .
[10] Mackay R. S. and Aubry S.,*Nonlinearity*, **7**(1994)1623 .
[11] Marín J. L. and Aubry S.,*Nonlinearity*,**9**(1996)1501.
[12] Kaneko K., *Physica D*, **34**(1989)1.
[13] Greenfield A., Putterman S. and Wright W.,*Phys. Lett. A*, **185** (1994)321.
[14] Tinkham M.,in *Introduction to Superconductivity*, (second edition) edited by McGraw-Hill, New York 1996.
[15] Aubry S., Flach S., Kladko K. and Olbrich E.,*Phys. Rev. Lett.*, **76**(1996)1607.
[16] Eilbeck J. C., Lomdahl P. S. and Scott A. C., *Physica D*, **16**(1985)318.
[17] Ott E.,*Chaos in Dynamical Systems*, edited by Cambridge Univ. Press 1993
[18] Watanabe S., Van der Zant H. S. J., Strogatz S. H. and Orlando T. P.,*Physica D*, **97**(1996) 429.
[19] In this respect, the analogy between discrete breathers and intrinsic defectibility (discommensurations) in modulated discrete elastic media (Floría L. M. and Mazo J. J.,*Adv. Phys.*,**45** (1996) 505) is by no means casual. Indeed, some of the most successful ideas and techniques used in recent studies on discrete breathers go back to earlier ones developed in connection with Frenkel-Kontorova models.
[20] Jian-Xue Xu and Jun Jiang, *Chaos, Solitons and Fractals*,**7**,(1996)3.
[21] Mackay R.S.,, in *Stochastic and Spatial Structures of Dynamical Systems*,, edited by S.J. van Strien and S.M. Verduyn Lunel. Ed. North-Holland. 1996.
[22] Dauxois T., Ruffo S., Torcini A. and Cretegny T., cond-mat 9709204 Preprint, 1997.
[23] Bonart D. and Page J., preprint.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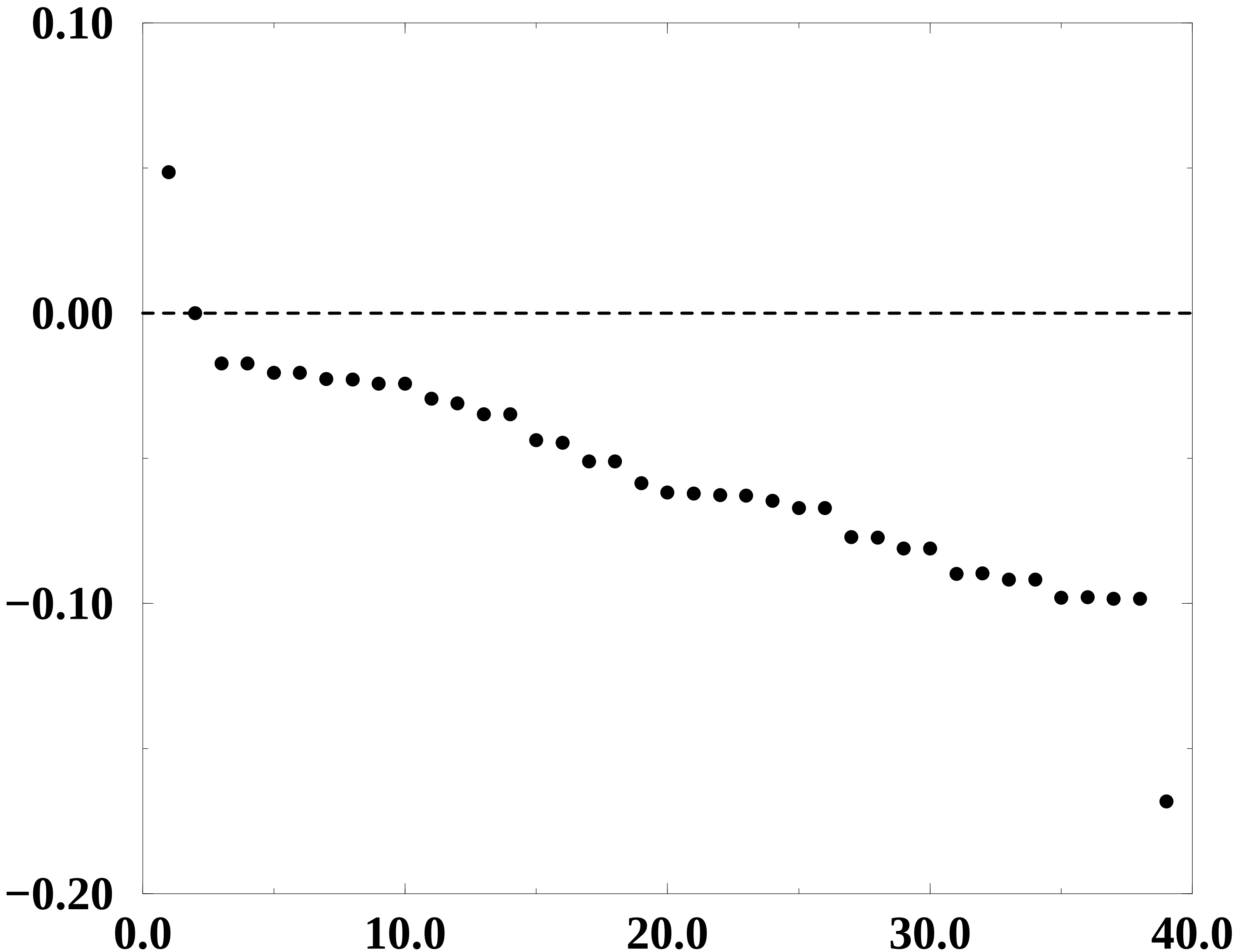
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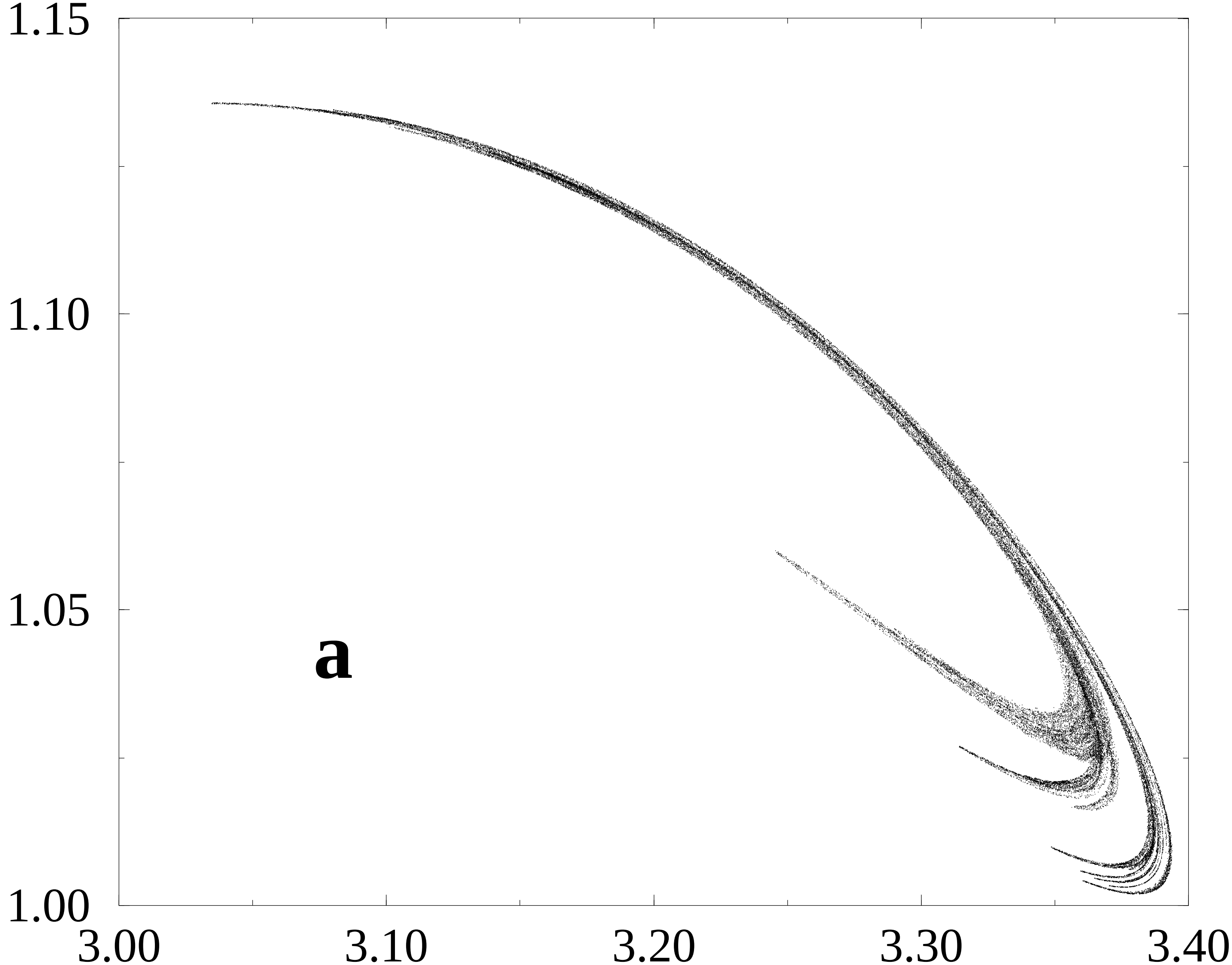

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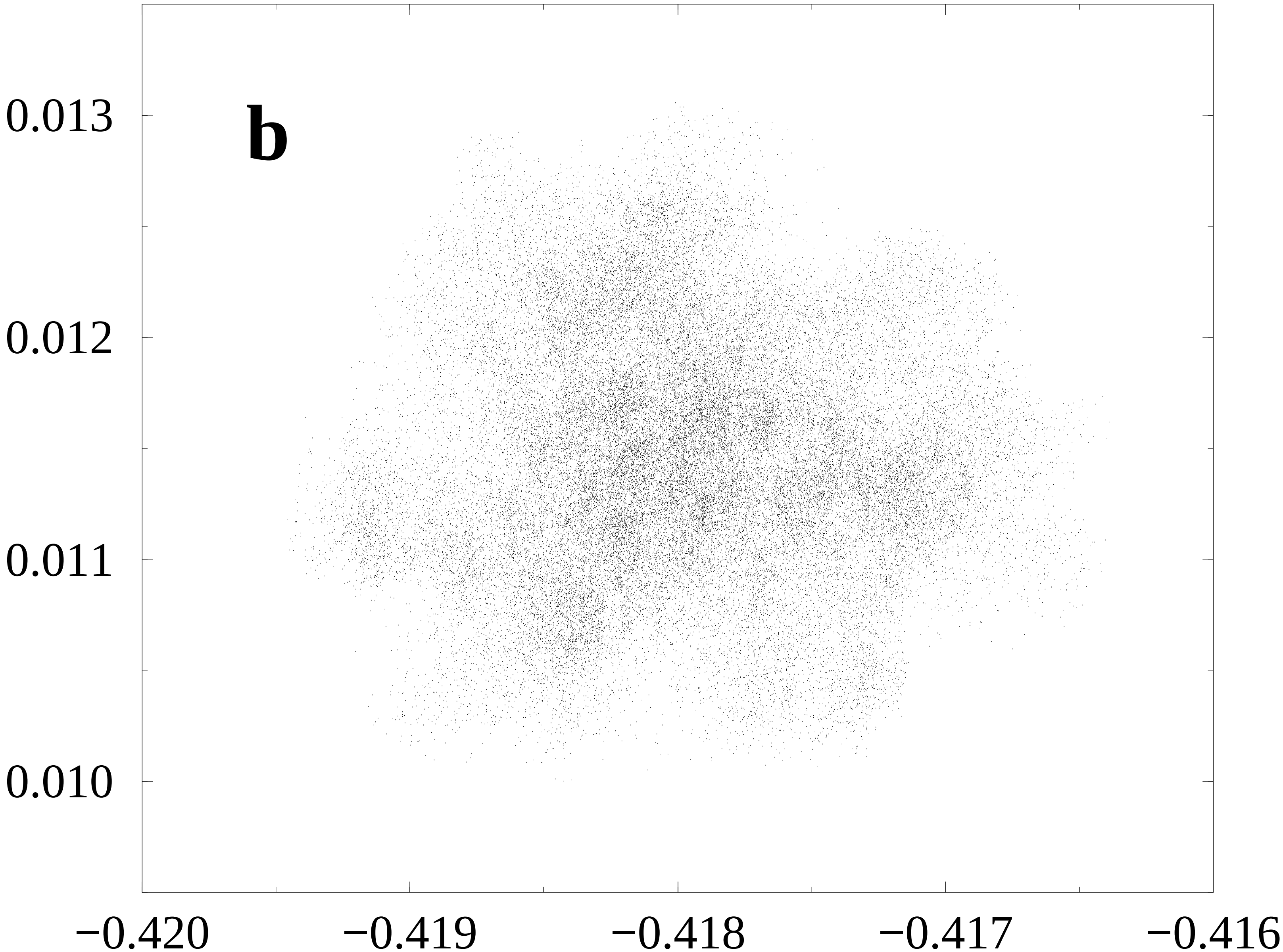

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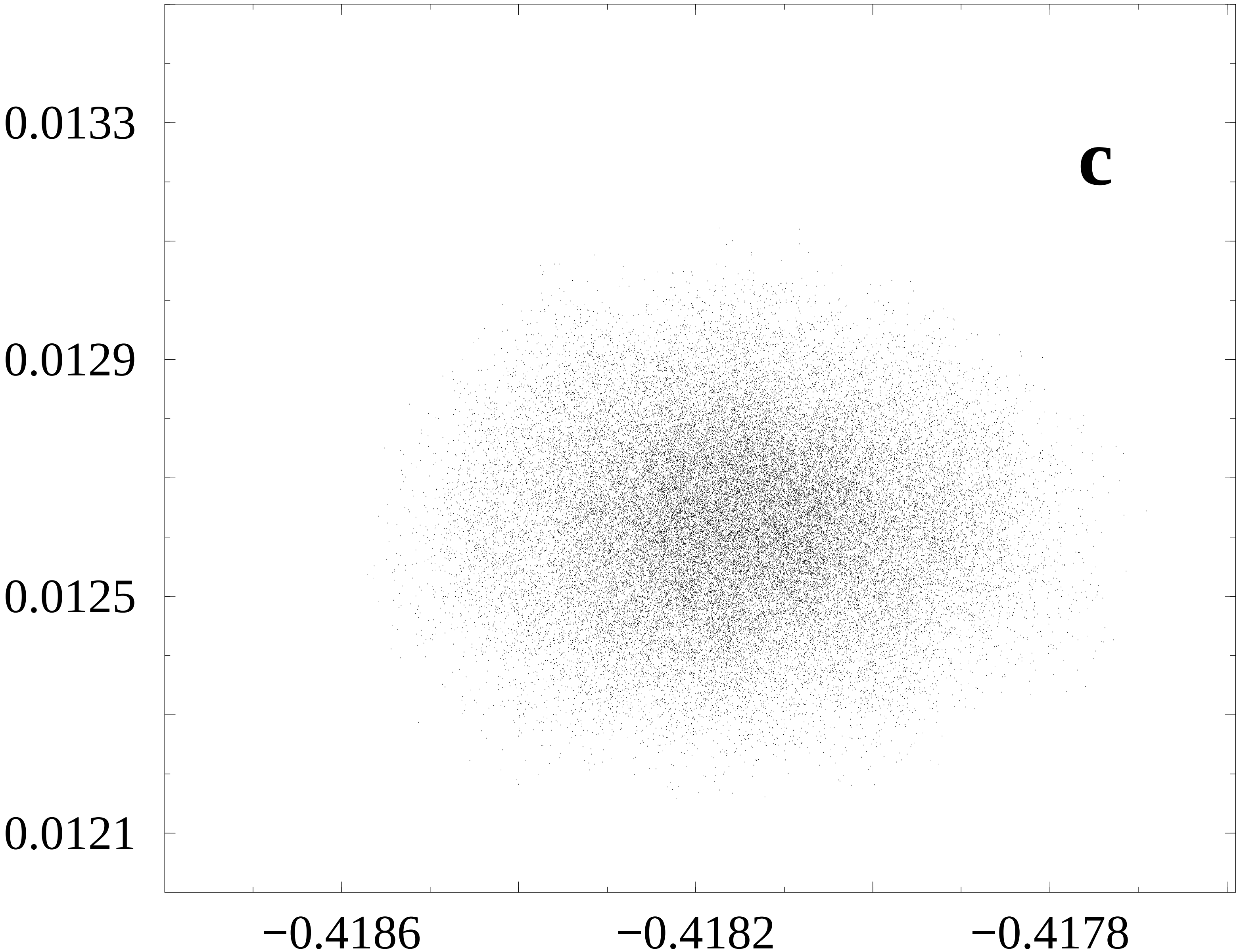
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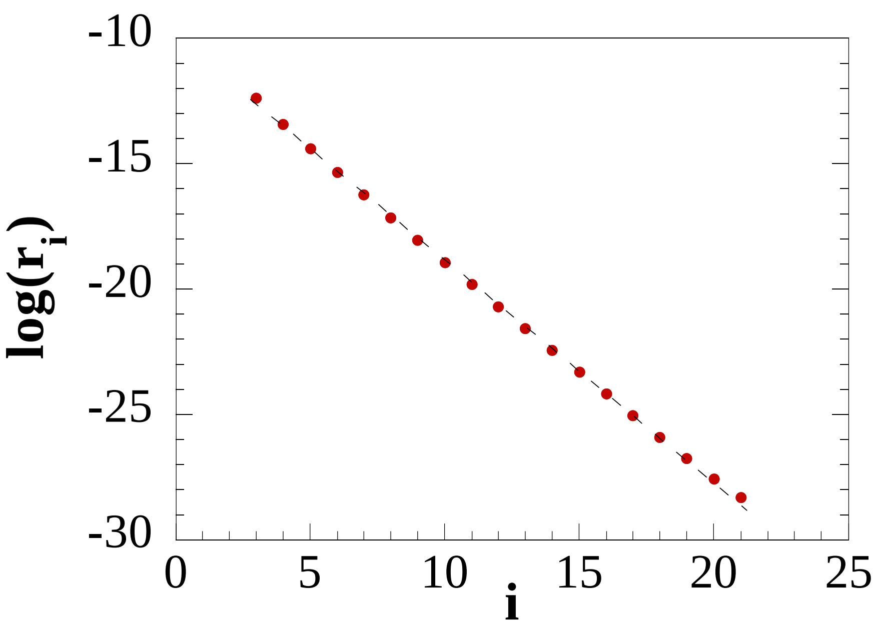

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